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16. ABSTRACT

This report discusses investigations of several problems of gravitation, including some aspects which are related to the Stanford University/Marshall Space Flight Center gyroscope experiment. In addition, the question of the existence of black holes is considered. While black holes like those in Einstein's theory may not exist in other gravity theories, trapped surfaces implying such black holes certainly do. The theories include those of Brans-Dicke, Lightman-Lee, Rosen, and Yang. A similar two-tensor theory of Yilmaz is investigated and found inconsistent and nonviable. The Newman-Penrose formalism for Riemannian geometries is adapted to general gravity theories and used to implement a search for twisting solutions of the gravity theories for empty and nonempty spaces. The method can be used to find the gravitational fields for all viable gravity theories. The rotating solutions are of particular importance for strong field interpretation of the Stanford/Marshall gyroscope experiment. Inhomogeneous cosmologies are examined in Einstein's theory as generalizations of homogeneous ones by raising the dimension of the invariance groups by one more parameter. The nine Bianchi classifications are extended to Rosen's theory of gravity for homogeneous cosmological models.

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GRAVITATION

I. INTRODUCTION

During the period June 6 to August 19, 1977, investigations were conducted by the author in the Cryogenic Physics Branch of the Space Physics Division of the Space Sciences Laboratory, Marshall Space Flight Center. This branch is involved in preparing the Stanford gyroscope experiment to detect the gravitational induction fields generated by rotating bodies [1]. These investigations were, therefore, in the general area of consideration of gravitational fields predicted by various currently viable, or proposed, theories of gravitation.

The question of the existence of black holes was treated with the aid of the symbolic manipulation language MACSYMA in collaboration with Dr. Richard Pavelle of Logicon Inc., a consultant to Hanscomb Fields's Air Force Cambridge Research Laboratories and to the Redstone Arsenal. Use of the term "black hole" will be reserved for later discussion, but trapped-surfaces [2], photon capture orbits, and photon capture impact parameters [3] were indeed found to exist in the gravitation theories of Einstein [3], Brans-Dicke [4], Rosen [5], Lightman-Lee [6], and Yang [7]. These properties strongly imply the existence of black holes in any gravitation theory with a Riemannian geometry.

A two-tensor theory of Yilmaz [8-10] with scalar field was completely and exhaustively studied. It was determined to be inconsistent, noncovariant, and completely devoid of gravitational effects. An earlier argument of Kraichnan [11] indicates that the result should be true for a wide class of gravitation theories.

Anisotropic cosmological models were investigated in Rosen's theory [12,13] of gravitation. It was found that the simple cosmologies become isotropic much more rapidly than in Einstein's theory. The nine Bianchi classifications have been extended to Rosen's theory for computations in the spatially homogeneous models.

Inhomogeneous cosmological models were investigated in the Einstein theory. The approach was to generalize spatially homogeneous models by raising the dimension of the invariance group by one parameter in the reverse of Inonu-Wigner's contraction [14]. Three cases are considered:

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a. The introduction of a peculiar velocity into self-similar cosmologies [15] to see what the major changes in observations would be (the generalization of Bianchi-Type VIII to the inhomogeneous case).

b. The generalization of Class III locally rotationally symmetric universes [16] to inhomogeneous spaces after a method suggested by Wainwright [17].

c. The extension of spatially homogeneous universes to inhomogeneous ones by adding an inhomogeneous nonsynchronous term to the metric.

The Bianchi groups operate on observer rest spaces which slice through the inhomogeneous space sections. This is a continuation of earlier work by the author [18] and is a view of cosmologies which has been discussed in a different form by Spero and Baierlein [19].

A solution has been found for cosmological models in Yang's theory of gravitation which is not an Einstein space. The observational consequences are straightforward to investigate since the metric is similar to that of the steady-state spatially homogeneous cosmological models.

The Newman-Penrose formalism [20] for Riemannian geometries allows one to investigate space times characterized by the presence or absence of one property or another. The use of the formalism to find solutions to any viable gravitation theory is being investigated. The relevant property is that the gravitational field propagate on twisting null rays. Thus, it is hoped that the means may be found for obtaining solutions appropriate to the exterior of a rotating star and to universes containing circularly polarized gravitational waves. Progress in these areas is outlined in this report.

II. EXISTENCE OF BLACK HOLES IN GRAVITATION THEORIES

Black holes have been an accepted feature of general relativity for several years, but only within the last 10 years have they become of widespread interest in gravitation and astrophysics [3]. Because they contain a singularity of zero volume and infinite matter density and, therefore, represent regions from which light cannot escape, they are a pathology of the theory. Indeed, many relativists have come to regard their existence as an indication of a flaw in general relativity. They believe that black holes have become a symptom of an intrinsic disease in general relativity itself for which there is no cure.

This situation has led, in some cases, to attempts to find alternative theories of gravity which will not predict the existence of black holes [5,6]. It has also sparked the hope that a proper marriage of general relativity with quantum field theory would prevent black-hole formation in an evaporative mass-loss process [21]. The mass-loss rate would conceivably match the collapse rate in such a way as to avoid the formation of the black hole.

This section will show that the same situation exists in all presently known viable gravitation theories. The criteria for the existence of black holes will be presented and then applied to these gravity theories. Examination of the two-dimensional collapse scenario with scalar quantum field theory leads finally to the conclusion that black-hole evaporation will be predicted by all these theories, implying no distinction between them and general relativity.

Let a gravitation theory define matter trajectories by the geodesic equation

$$\ell_{a\,\|\,b}\,\ell^{b} = 0$$

where $\|$ indicates covariant differentiation and ℓ^{a} is tangent to the trajectory. Then a congruence of trajectories will obey the geodesic deviation equation relating the connecting vector n^{a} to the Riemann tensor $R^{a}{}_{bcd}$:

$$\left(n^{a}{}_{\|\,e}\,\ell^{e}\right)_{\|\,f}\,\ell^{f} = R^{a}{}_{bcd}\,\ell^{b}\,n^{c}\,\ell^{d} \quad .$$

Let these trajectories be embedded in a Riemannian manifold on which the Bianchi identities hold. Then the effects of the deviation equation will be contained in the Newman-Penrose equations [20]. These equations connect the evolution of the geometrical spin coefficients to each other. The spin coefficients, in turn, contain the information about the physical behavior of the matter trajectories' congruences.

The spin coefficient of interest is ρ, the complex expansion of the congruence, describing the expansion

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$$\theta = \ell^{a}{}_{\|a}$$

and the rotation

$$\omega = \left(\ell_{[a\|b]}\ell^{[a\|b]}\right)1/2$$

of the rays (the bracket, [ ], denotes antisymmetrization)

$$\rho = \theta + i\omega \quad .$$

Let ℓ_a be tangent to an outgoing null geodesic and n_a tangent to an ingoing null geodesic. Let the complex vector m_a span the celestial sphere

$$m_a = \hat{e}_2 + i\hat{e}_3 \quad ,$$

where $\hat{e}_2$ and $\hat{e}_3$ are unit vectors and the celestial sphere is treated as an almost-complex two-dimensional manifold. The set of vectors forms a complex null tetrad from which the metric g_{ab} is composed:

$$g_{ab} = 2\ell_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)} \quad ;$$

$$\ell_a n^a = n_a \ell^a = -m_a \bar{m}^a = -\bar{m}_a m^a = -1$$

(all other inner products vanish; the parenthesis, ( ), denotes symmetrization and the overbar is complex conjugate). Then

$$\rho = \ell_{a\|b} m^a \bar{m}^b \quad ,$$

and the Newman-Penrose equation is

$$\rho_{\|a} \ell^a = -\rho^2 - \sigma\bar{\sigma} - R^c{}_{acb} \ell^a \ell^b \quad ;$$

σ is the shear:

$$\sigma = \ell_{a\|b} m^a m^b \quad .$$

For real ρ, this indicates that $d\theta/ds \sim -\theta^2 - \sigma^2 - R$ where s is proper time along the geodesic, implying that once $\theta < 0$ it is always so. For a static metric, the region wherein $\theta < 0$ is bounded by a "trapped surface" within which all geodesic congruences converge to a singularity of zero volume, infinite matter density, and infinite tidal forces [2]. This is a black hole.

We consider the static spherically symmetric metrics in isotropic form

$$ds^2 = -e^{2\phi} dt^2 + e^{2\psi} (dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2) \qquad (1)$$

where

$$\phi = \phi(R)$$

and

$$\psi = \psi(R) \quad .$$

We use the isotropic form since most closed-form solutions for various gravity theories are given the isotropic form. We translate to a luminosity (retarded-time) coordinate via

$$e^{\phi}du = e^{\phi}dt - e^{\psi}dR$$

to find

$$ds^2 = -e^{2\phi}du^2 - 2e^{\phi+\psi}dudR + e^{2\psi}R^2d\Omega^2 \quad , \tag{2}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad .$$

From the preceding discussion we find

$$\rho = (1 + \psi'R)/R \quad , \tag{3}$$

where $\psi' = \partial\psi/\partial R$. Since there is no rotation, $\rho = \theta$ and $\theta < 0$ if $1 + \psi'R < 0$, where $\theta = 0$ at the trapped-surface radius R_t. This is clearly so if ψ can be represented as an inverse power series in R,

$$\psi = \sum a_n(1/R)^n \quad ,$$

for integer n and non-negative constants a_n.

Examining geodesics, we must see if particles in orbit or incident from a great distance are captured by the hole, because a black hole must consume anything that comes too close. For metric (1) the geodesic equations with motion in the equatorial plane give

$$g_{\Phi\Phi}\ell^3 = h$$

and

$$g_{00}\ell^0 = E \quad ,$$

two constants of the motion defining the impact parameter $\lambda = E/h$. We can then arrive at the equation

$$dR/d\phi = \pm(R/\lambda)\,[R^2 d^{2(\psi-\phi)} - \lambda^2 + (\epsilon^2\lambda^2 e^{2\psi} R^2)/h^2]^{1/2} \tag{4}$$

where $\epsilon = 0, -1$ for null and timelike ℓ_a. Because material particles usually have a more difficult time of it than the photons, we need only look at $\epsilon = 0$. Then,

$$dR/d\Phi = \pm(R/\lambda)\,[R^2 e^{2(\psi-\phi)} - \lambda^2]^{1/2} \quad . \tag{5}$$

Orbits are stable down to a critical radius R_c gound by solving $dR/d\Phi = 0$. The capture impact parameter is λ_c, found from $d/dR(dR/d\Phi) = 0$. For R_c and λ_c this gives

$$1 + R(\phi' + \psi')\,\Big|_{R=R_c} = 0 \tag{6}$$

and

$$\lambda_c = R_c \exp\{2[\phi(R_c) - \psi(R_c)]\} \quad . \tag{7}$$

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The results are given in Table 1. The results for the Reissner-Nördstrom metric in Einstein's theory are presented first. Listed across the table are the theory designations $e^{2\phi}$, $e^{2\psi}$, R_t, and r_t, the trapped-surface radii in isotropic and Schwarzschild coordinates

$$ds^2 = -e^f dt^2 + e^g dr^2 + r d\Omega^2$$

the two capture radii R_c (isotropic) and r_c (Schwarzchild), and the coordinate-independent λ_c.

This obviously disproves the previous claims that black holes do not exist in Rosen's theory [5] or are not approachable by a geodesic [22] in the Lightman-Lee theory [6]. Black holes are, indeed, found to exist in the Kilmister-Yang theory [7]. In another solution for that theory [23],

$$e^{2\phi} = e^{2\psi} = (1 - m/R)^2$$

and

$$\rho = R/(R - M) \quad ,$$

which is an impenetrable barrier at $R = M$ or $r = 0$. The Brans-Dicke [4] black-hole surface has imaginary radius unless $\omega < 0$, which is a viable form of the theory [24]. This disproves all prior claims that Brans-Dicke black holes are like those of general relativity, "Schwarzschild" [25]. Clearly, all these theories predict static black holes, whether or not their proponents have claimed so.

For black-hole evaporation we follow the discussions of Davies, Fulling, and Unruh [26]. We choose the scalar field in a two-dimensional space time (of signature zero) because the quantum field theory is solvable. The metric of such a space time may always be written in a conformally flat form

TABLE 1. RESULTS ON BLACK-HOLE EXISTENCE FOR SEVERAL PRESENTLY KNOWN VIABLE GRAVITATION THEORIES

	e^{2b}	e^{2c}	R_t	r_t	R_c	r_c	A_c
Einstein, Reissner-Nordstrom E = electric charge $K = (9M^2 - 4E^2)^{1/2}$ $M = K = 3M$	$\left[\frac{1-[(M^2-E^2)-4R^2]^{1/2}}{\left(1-\frac{M-E}{2R}\right)\left(1-\frac{M-E}{2R}\right)}\right]^2$	$\left(1-\frac{M-E}{2R}\right)^2\left(1-\frac{M-E}{2R}\right)^2$	$\frac{1}{2}(M^2-E^2)^{1/2}$	$M-(M^2-E^2)^{1/2}$	$\frac{1}{4}(M-K)\left[1+\frac{2(3M-K)^{1/2}}{(M-K)^{1/2}}\right]$	$\frac{3M}{2}-\frac{1}{2}(9M^2-4E^2)^{1/2}$	$\frac{(3M-K)^{3/2}}{\sqrt{2}(M-K)^{1/2}}$
Einstein E = 0,	$\frac{\left[1-(M/2R)\right]^2}{\left[1-(M/2R)\right]^2}$	$\left(\frac{M}{2R}-1\right)^4$	$\frac{M}{2}$	$2M$	$M\left(1-\frac{\sqrt{3}}{2}\right)$	$3M$	$3\sqrt{3}M$
Rosen $\delta = 2, 4, \ldots$	$e^{-2M/r}$	$e^{2M/R}$	M	eM	$2M$	$2\sqrt{eM}$	$2eM$
Yang	1	$\left(1-\frac{M}{2R}\right)^4$	$\frac{M}{2}$	$2M$	$\frac{M}{2}$	$2M$	$2M$
Lightman-Lee	$\frac{[1-(M/2R)]^2}{[1-(M/2R)]^2}$	$\frac{[1-(M/2R)]^2}{[1-(3M/2R)]^2}$	$\frac{M}{2}(3-\sqrt{6})$	$\frac{M}{2}(5-2\sqrt{6})$	$\frac{M}{2}(5-2\sqrt{3})$	$\frac{M}{2}(5-3\sqrt{3})$	$\frac{M}{2}(7-4\sqrt{3})$
Brans-Dicke (See Reference 12 for metric components.)	$p \quad \frac{\sqrt{2}(\omega-1)}{\sqrt{\omega-2}\sqrt{2\omega-3}}$ $\frac{M\sqrt{\omega-2}}{\sqrt{2}\sqrt{2\omega-3}}$	$\frac{M}{2\omega-3}\left(1-\omega+\sqrt{2\omega-3}\right)$	$\frac{(R_t-\sigma)^{1-p}(R_t-\sigma)^{1-p}}{R_t}$		$M\left(1-\frac{\sqrt{3\omega-4}}{\sqrt{4\omega-6}}\right)\frac{(R_c-\sigma)^{1-p}(R_c-\sigma)^{1-p}}{R_c}$		$\frac{(R_c-\sigma)^{1-p-2c}M}{R_c(R_c-\sigma)^{-1-p-2c}M}$

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$$ds^2 = C(u,v)\, du dv \quad ,$$

where

$$u = t - \int e^{\psi - \phi}\, dR \quad ,$$

$$v = t + \int e^{\psi - \phi}\, dR \quad ,$$

and

$$C(u,v) = e^{2\phi} \quad .$$

Using a geodesic point-splitting renormalization procedure, the proponents find a satisfactory form for the stress-energy tensor. There are several coordinate/regularization-dependent parts of

$$T_{ab} = g_{ab} R^{dc}{}_{dc} / 48\pi \pm g_{ab} / 8\pi\epsilon^2 + \theta_{ab}$$

which represent vacuum polarization contributions, where $R_{dc}{}^{dc}$ is the curvature and ϵ is the point-splitting parameter. The first two terms may be regarded as hydrodynamic and cosmological constant contributions;

$$\theta_{ab} = -(12\pi)^{-1} C^{1/2} \partial^2 (C^{-1/2}) / \partial x^a \partial x^b$$

is the vacuum-polarization/scalar particle-creation contribution.

The appearance of radiation depends on the boundary conditions imposed on the coordinate system at infinity [27,28]. For the collapse situation, this means that the surface at asymptotically flat infinity must accelerate from the surface of a thin shell of matter which provides the mass M in the metric

$C(u,v)$ (equivalent to the shell accelerating away from that surface as it collapses). This implies a coordinate transformation

$$u = K\ell n(A - \bar{u})$$

where K and A are constants depending on the mass and the particular collapse dynamics. We then use

$$ds^2 = C(\bar{u},v)\,d\bar{u}dv$$

in the T_{ab} computation. Because such boundary conditions may be found for any time-like 2-surface [26], the collapse T_{ab} can be determined by the same formula for all the gravity theories previously discussed. We have computed θ_{ab} and found a constant Hawking flux term of the form $\sim 1/K^2$ for all the metrics discussed. It extracts mass from the black hole as particles are radiated to infinity [27]. Thus, black-hole evaporation is very likely a common feature of all gravitation theories.

This establishes that black holes are a normal, not pathological, feature of a viable gravitation theory. Also, when gravity, thermodynamics, and quantum field theory are properly married (menage a trois?), static black holes evaporate, radiating particles. Such a process may even prevent their formation [21]. Thus, it is useless to use the existence and evaporation of black holes as a test to determine the relative viability of gravitation theories.

III. NONVIABILITY OF THE TWO-TENSOR PLUS SCALAR GRAVITY THEORY OF YILMAZ

A number of scalar-tensor [29] and two-tensor [30,31] gravitation theories are currently viable. In certain of the two-tensor theories, the trace of one of the tensors plays the role of a scalar field. Alternatively, one could treat the scalar field of a scalar-tensor theory as the trace of a second tensor and generalize the theory to include the second tensor. One such theory [8] and its generalizations [9,10] are analyzed in this section. The results of such

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generalizations are shown to pose severe problems of consistency, uniqueness, covariance, and coupling of gravitational fields to matter so that gravitational effects exist. Thus, unfortunately, we find that this theory and its generalizations are not viable.

The 1955 theory [8] leads to the static metric

$$ds^2 = e^{2M/r}(dr^2 + r^2 d\phi^2) - e^{-2M/r} dt^2 \quad .$$

It is essentially a scalar-tensor theory. The results satisfy the three classical tests. Page and Tupper corrected a Langrangian error for the theory and showed that the corrected theory possesses non-unique field equations and that the static metric given previously is not the unique spherically symmetric static solution. A similar theory of Papapetrou has been shown nonviable. Ni has shown that it also is in violent conflict with a few experiments, including the perihelion shift. The subsequent development of the theory to a two-tensor theory has failed to correct these problems.

The second tensor h_{ab} generated from the scalar as its trace is brought into the theory [9,10] via a local Lorentz invariance argument and related to the stress energy tensor via the equation

$$h_{ab} \| a^{\| a} = 4\pi T_{ab}$$

where $\|$ is the covariant derivative. The metric g_{ab} is then found from the relation

$$dg_{ab} = 2\,(g_{ab}dh - g_{ai}dh^{i}_{b}) \quad .$$

A kind of Einstein field equation is then given by

$$G_{ab} = 8\pi\,(T_{ab} + t_{ab}/4\pi)$$

where t_{ab} is the energy tensor for the gravitational field h_{ab}. These terms are as given in the references which report the development of the theory [9,10]. A gauge condition

$$h^{a}_{b|a} = h^{a|b}_{b} = 0$$

leads to

$$t_{ab} = -2\left(h^{c}_{d|a}h^{d}_{c|b} - \frac{1}{2}g_{ab}h_{cd|e}h^{cd|e} + h_{|a}{}^{h}{}_{|b} - \frac{1}{2}g_{ab}h_{|c}h^{|c}\right).$$

The geodesic equations are imposed to describe matter trajectories, as is usual in attempts to formulate a gravitation theory embedded in a Reimannian manifold. One finds

$$du_j/ds = \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} u_i u^k \quad , \quad u_j = dx_j/ds \quad .$$

The metric $\tilde{g}$ is related to the Lorentz metric $\tilde{h}$ and second tensor $\tilde{h}$ by

$$\tilde{g} = \tilde{n} \cdot \exp[2(h\tilde{I} - 2\tilde{h})] \quad .$$

The metric $\tilde{g}$ to third order in $\tilde{h}$ is then

$$g_{ij} = \frac{4}{3}\left(h^{3}n_{ij} - 6h^{2}h_{ij} + 12hh_{i}^{a}h_{aj} - 8h_{i}^{a}h_{ab}h_{j}^{b}\right) + n_{ij} + 2(hn_{ij} - 2h_{ij})$$

$$+ 2\left(h^{2}n_{ij} - 4h_{ij}h + 4h_{i}^{a}h_{aj}\right) \quad .$$

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Questions of uniqueness, self-consistency, and actual predicted effects are examined in this theory. First, the field equations involving the Einstein tensor differ in third order from the previously cited expansion. In fact, we find on substituting it into the field equation that the field equations do not imply the relation between $\tilde{g}$, $\tilde{n}$, and $\tilde{h}$ given previously to third order. Therefore, the theory's consistency ends there.

Second, in looking for useful approximations, attempts have been made to iterate the field equation to third and higher order to prove integrability of the field equation. For parameter $\lambda \ll 1$, we substitute

$$g_{ij} = n_{ij} + \lambda p_{ij} + \lambda^2 k_{ij} + \lambda^3 J_{ij} + O(\lambda^4)$$

into the field equations and find

$$g_{ij} = n_{ij}(1 + 2h + 2h^2) - 4h_{ij} - 8hh_{ij} + 8n_{ab}h_i^a h_j^b$$

up to second order consistently for p_{ij} and k_{ij}. But we also find

$$J_{ij} = \frac{4}{3}\left(h^3 n_{ij} - 6h^2 h_{ij} + 12hh_i^a h_{aj} - 24\int h_i^a h_j^b dh_{ab}\right) ,$$

showing that the equations are not integrable to third order. Thus, the equations can only be consistent and integrable to second order in h_{ab}. However, one can go further by rewriting the field equations

$$G_{ij} - 2h_{ij\,\|\,a}^{\quad\|\,a} - 2t_{ij} = E^a_{ij|a} + O(h^3) = 0$$

$$E^a_{ij} = -8\left[h\left(h^a_{(i|j)} + h^{ab}\left(h_{b(i|j)}\right) + h^a_{b|(i}h^b_{j)} - h^b_{(i}h^a_{j)|b}\right)\right] .$$

To verify, the gauge conditions must be used frequently. $E^{a}{}_{ij|a}$ does not vanish; therefore, the field equations are themselves inconsistent to second order.

Considering t_{ab} and using the gauge conditions, we find a large portion of t_{ab} vanishes. Therefore, it is not the energy density of a spin-two field and $t^{i}_{j|i} \neq 0$; consequently, Noether's theory is violated. At this stage it is clear that this is at best second order theory. Continuing, we use the h_{ab} field equation to write the geodesics as

$$p\, du_i/ds = p\left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\} u_j u^k = (4\pi)^{-1}\left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\} h_j{}^{k\,\|\,a}{}_{\|\,a}$$

where p is the matter density. The Christoffel symbol to first order is

$$\left\{ \begin{matrix} a \\ i\ j \end{matrix} \right\} = 2\left(2\delta^{a}{}_{(i}h_{|j)} - n_{ij}h^{|a} - 2h_{ij}{}^{|a} - 4h^{a}_{(i|j)}\right) .$$

Insertion of this into the geodesic equation, partial integration, and use of the gauge conditions consistently leaves

$$\left\{ \begin{matrix} j \\ i\ k \end{matrix} \right\} u_j u^k = 0 ,$$

and the equation of motion is

$$d^2x_i/ds^2 = 0 .$$

Thus, there are no gravitational effects in the theory to all orders. All motion is simple inertial motion in the theory; therefore, there are no gravitational

effects. We conclude that one must be very careful in generalizing a scalar theory to a tensor theory. In particular, gauge conditions must be kept an auxiliary (not essential) part of the theory.

IV. HOMOGENEOUS ANISOTROPIC COSMOLOGICAL MODEL IN ROSEN'S GRAVITATION THEORY

Rosen's theory is a two-metric theory with prior geometry [5]. We shall consider one of the metrics to be the Lorentz metric $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$. The field equations in Rosen's theory for $g_{\mu\nu}$ are

$$N_{\mu\nu} - \frac{1}{2} N g_{\mu\nu} = -8\pi k\, T_{\mu\nu}$$

where

$$N_{\mu\nu} = \frac{1}{2}\left(g^{\nu\lambda} g_{\mu\lambda|\alpha}\right)|\alpha$$

and

$$k = (-g)^{1/2} \quad .$$

The slash (|) indicates the covariant derivative with respect to $\gamma_{\mu\nu}$, and $T_{\mu\nu}$ is the stress density tensor.

Since $g_{\mu\nu}$ is a Riemannian metric, we adopt the Bianchi classification scheme to the homogeneous three spaces [32]. We also wish to express the field equations in terms of the irreducible parts of the stress tensor which possess invariant physical significance. Hence, one desires to have equations for T_{oo}, T_{oi}, T^k_k, and $T^i_j - \frac{1}{3}\delta^i_j T^k_k$ (the Latin indices running 1-3).

If the metric is expressed in terms of a basis tetrad

$$g_{\mu\nu} = E_{\mu}^{A} E_{A\nu} \quad ,$$

then the affine connection coefficients become

$$\Gamma^{\mu}_{\sigma\tau} = E^{\mu}_{A|B} E_{\sigma}^{A} E_{\tau}^{B} \quad ,$$

and we have

$$g_{\mu\nu|\alpha} = \Gamma^{\nu}_{\mu\alpha} + \Gamma^{\mu}_{\nu\alpha} \quad .$$

Then Rosen's tensor $N_{\mu\nu}$ may be constructed in terms of the affine connection,

$$N_{\mu}^{\nu} = \frac{1}{2} g^{\nu\lambda} \left(\Gamma^{\lambda}_{\mu\alpha|\alpha} + \Gamma^{\mu}_{\lambda\alpha|\alpha} \right) - \frac{1}{2} \left(\Gamma^{\lambda}_{\mu\alpha} + \Gamma^{\mu}_{\lambda\alpha} \right) \left(\Gamma^{\nu}_{\lambda\alpha} + \Gamma^{\lambda}_{\nu\alpha} \right) ,$$

and the field equations written in terms of the connection coefficients implied by a particular set of symmetries chosen for the spacetime. The Bianchi classifications may then be used.

A suitable metric would be given by the line element

$$ds^2 = e^{2\Omega} dt^2 - e^{2\alpha} e^{2\beta}_{ij} E_{A}^{i} E_{B}^{j} dx^{A} dx^{B} \quad ,$$

where Ω, α, and β_{ij} are functions of the time. The metric is similar to a form widely used now in Einstein's theory [32].

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The Bianchi case

$$E_A^i = \delta_A^i$$

will now be considered. The dynamical field equations are then

$$-\frac{3}{2}\ddot{\alpha} - 6\dot{\alpha}^2 - 2\dot{\beta}_{ik}\dot{\beta}_{ki} = -8\pi T_{oo}$$

and

$$\ddot{\beta}_{ij} + 8\dot{\alpha}\dot{\beta}_{ij} + 4\left(\dot{\beta}_{ij} - \frac{1}{3}\delta_{ij}\dot{\beta}_{k\ell}\dot{\beta}_{k\ell}\right) = -8\pi\pi_j^i \quad ,$$

where π_{ij} are the trace-free stresses. For a diagonal β_{ij}, the fluid shear is

$$\sigma_{ij} = \dot{\beta}_{ij} \quad .$$

The shear field equation

$$\pi_{ij} = -\lambda\sigma_{ij} \quad ,$$

where λ = viscosity is then

$$d\sigma_{ij}/dt + 4\left(\sigma_{ik}\sigma_{kj} - \frac{1}{3}\delta_{ij}\sigma^2\right) = -8\pi\lambda\sigma_{ij} - 8\dot{\alpha}\sigma_{ij}$$

and for small shear ($\sigma^2 << 1$) we find

$$\sigma_{ij} = [\text{const.}]\, e^{-8\pi\lambda(t-t_o)}\, e^{-8(\alpha-\alpha_o)} \quad .$$

Anisotropy damping is much more rapid in Rosen's theory than in Einstein's theory.

For a dust source, the hydrodynamic conservation law gives the energy density as

$$\rho = \rho_o e^{-3\alpha} \quad ,$$

as in Einstein's theory. Then, we have

$$\frac{3}{2}\ddot{\alpha} + 6\dot{\alpha}^2 + 2\sigma_o^2 e^{-16\alpha} = -8\pi p_o e^{-3\alpha} \quad ,$$

and from the T^k_k equation $\ddot{\alpha}$ may be eliminated to find a Friedmann-like equation. The Bianchi classifications are being investigated in this context. The most interesting aspect of this is the isotropization question: Will Rosen's theory provide a better key to the isotropy of the universe?

V. INHOMOGENEOUS COSMOLOGICAL MODELS IN EINSTEIN'S THEORY

A. Tilted Self-Similar Cosmologies

Self-similar cosmologies are models that admit a group of similarity transformations; i.e., an invariance under changes of length scale [15]. There is defined a homothetic vector by the relation

$$\nabla_{(\mu}\xi_{\nu)} = \lambda g_{\mu\nu}$$

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where λ is a scalar and ∇_μ is a covariant derivative. Tilted models are those in which the fluid flow vector is tilted away from the time-like vector normal to the space-like hypersurfaces η_μ by an hyperbolic angle β [33]. We combine the two formalisms in search of interesting models. The particular question is: Are there tilted universes with submanifolds, invariant under the Bianchi I group (translations), with those manifolds submanifolded to ones invariant under a similar group and with the fluid flow shear free? Eardley has mapped out the self-similar formalism [15], and King and Ellis describe the methods needed for tilted models [33].

The equations of Eardley were modified to include a tilted flow vector and specialized to the case with Bianchi I submanifold. The undefined quantities are those given in Eardley's paper. The metric is

$$ds^2 = e^{2\psi}\left[-dz^2 + g_{ab}(z)\ \sigma^a \sigma^b\right]$$

where z labels invariant submanifolds and $d\psi = b_a \sigma^a$ (b_a is a vector related to the structure functions of the similarity group H_3). The σ^μ generate a G_2 subgroup characterized by the Bianchi classification, and

$$g_{ab}(z) = e^{2\alpha} e^{2\beta}_{ab} \quad ,$$

$$\alpha = \alpha(z) \quad ,$$

$$\beta_{ab} = \beta_{ab}(z) \quad ,$$

and

$$\beta^a_a = 0 \quad .$$

For Bianchi I submanifolds invariant under G_2 (the field with shear-free flow), the field equations are:

$$-3\dot{\alpha}^2 + b^2 = T^o_o e^{-2\psi} \quad ,$$

$$-2b_k\dot{\alpha} = T^o_k e^{-2\psi} \quad ,$$

$$-6\ddot{\alpha} - 9\dot{\alpha}^2 + 3b^2 = T^k_k e^{-2\psi}$$

and

$$2b_i b_\ell - \frac{2}{3}\delta_{i\ell} b^2 = \pi_{i\ell} e^{-2\psi}$$

where

$$\dot{x} = dx/dz \quad .$$

Reducing $T_{\mu\nu}$ to its irreducible parts, a perfect fluid source,

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu} \quad ,$$

is obtained. With further specialization, the field equations and fluid conservation laws (shear free $\rightarrow \beta_{ab} = 0$) are:

$$-3\dot{\alpha}^2 + e^{-2\alpha}\, b_i b^i = -\left[(\rho + p)\, u_o^2 - p\right] e^{-2\psi} \quad ,$$

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$$-2e^{-\alpha} b_k \dot{\alpha} = -e^{-2\psi} [(\rho + p) u_k u_o] \quad ,$$

$$-6\ddot{\alpha} - 9\dot{\alpha}^2 + 3e^{-2\alpha} b_i b^i = 3pe^{-2\psi} \quad ,$$

$$\left[b_i b_\ell - \frac{1}{3} \delta_{i\ell} b^k b_k \right] e^{-2\alpha} = \frac{e^{-2\psi}}{2} \left[(\rho + p) u_i u_\ell - \frac{1}{3} \delta_{i\ell} u_k u^k \right] \quad ,$$

$$\dot{\rho} + 3(\rho + p) \dot{\alpha} = 0 \quad ,$$

$$(\rho + p) \dot{u}_o = u_k u^k p_{|o}$$

and

$$(\rho + p) \dot{u}_k = -u_k u^o p_{|o} \quad .$$

The equation of state is $p = \gamma\rho$ and the equation for flow normalization is $u_\mu u^\mu = -1$.

For the hydrodynamic equations,

$$\rho = \rho_o \exp[-3(1 + \gamma) \alpha]$$

and

$$u_a = \tilde{u}^o_a \exp[-3\gamma\alpha] \quad ,$$

and for the trace-free stress equations

$$b_i = \left[\tilde{u}{}^{o}_{i} (1 + \gamma)\, \rho_o / 2 \right]^{1/2} e^{-\psi}\, e^{-3(1 + 3\gamma)\, \alpha/2} \quad .$$

The Friedman-like equation for $\dot{\alpha}$ is solved after the substitution $\alpha = \ell n\, a$ to give

$$\frac{2a^{(3\gamma + 1)/2}}{(3\gamma + 1)} \Bigg|_{a_o}^{a} = x_o \int_{z_o}^{z} e^{-\psi}\, dz \quad .$$

The density, ρ, can be arbitrarily specified as a function of the space coordinates at time z_o, $\rho_o = \rho_o(x)$. Since these are self-similar models [34], $z = t^n x^m$, where n and m are real numbers and x is any space coordinate. The function ψ is fixed by the condition $\langle d\psi | \nabla z \rangle = 0$ [15].

Since

$$d\psi = b_a \sigma^a = b_a dx^a \quad ,$$

suppose we choose $b_3 \neq 0$; then we do, indeed, have the solution

$$3a^{(3\gamma + 1)/2} / (3\gamma + 1) = x_o\, e^{-x_3}\, t^n x_2^m \quad .$$

B. Universes with Nonsynchronous Time

Now, let the metric be of the form

$$ds^2 = -\left(dt - u_a z^a \right)^2 + \delta_{ij} z^i z^j \quad ,$$

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where the forms

$$z^a = A(t)\, B^a_j(t)\, E^j_i\, dx^i$$

$$= e^\alpha\, e^\beta_{aj}\, \sigma^j \quad ,$$

and where

$$u_a = u_a(t,x) \quad ,$$

is a function of time and the space coordinates.

In this system, the velocity form is

$$\hat{u} = -z^o \rightarrow u_a = -\delta^o_a$$

and the "time" coordinate is

$$dx^o = dt - u_a z^a \quad .$$

This makes a very convenient system for computations; i.e., the projection operator is simply

$$h^\nu_\mu = \delta^\nu_\mu + \delta^o_\mu\, \delta^\nu_o \quad .$$

The convective derivative, $u^a \nabla_a$, is given as ∇_o and, in the case of scalars, is given as $\partial/\partial x^o$. The kinematical gradients of the fluid velocity are also

easy to compute since the Γ^{i}_{jk} (purely spatial) rotation coefficients do not appear in the computations when $u_a = -\delta^{o}_{a}$.

In terms of acceleration a_i, expansion S_{mr}, rotation W_{mr}, reference frame spin t_{mr}, and group structure functions $C^{j}_{k\ell}$, we find

$$\Gamma^{i}_{oo} = a_i \quad ,$$

$$\Gamma^{m}_{ro} = S_{mr} + W_{mr} \quad ,$$

$$\Gamma^{r}_{om} = -t_{mr} \quad ,$$

$$\Gamma^{i}_{jk} = \bar{C}^{j}_{ki} + t_{ik}u_j \quad ,$$

and

$$\bar{C}^{j}_{ki} = \frac{1}{2}\left(C^{j}_{ik} + C^{k}_{ji} - C^{i}_{kj}\right)$$

for the rotation coefficients. In terms of the metric functions, these kinematical quantities are:

$$S_{mr} = \dot{\alpha}\delta_{mr} + \sigma_{mr} + u_{(m|r)} + \dot{u}_{(m}u_{r)}$$

$$W_{mr} = u_{[m|r]} + \dot{u}_{[m}u_{r]} + u_j t_{j[m}u_{r]} + u_k \bar{C}^{k}_{mr}$$

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and

$$a_i = \dot{u}_i + S_{ij}u_j$$

where

$$\sigma_{mr} = \dot{B}_{k(m}B^{-1}_{r)\,k} = (e^{\beta})^{\cdot}_{k(m}(e^{-\beta})_{r)\,k} \quad ,$$

$$t_{mr} = \dot{B}_{k[m}B^{-1}_{r]\,k} = (e^{\beta})^{\cdot}_{k[m}(e^{-\beta})_{r]k} \quad ,$$

and

$$t_{mr} = \dot{B}_{k[m}B^{-1}_{r]\,k} = (e^{\beta})^{\cdot}_{k[m}(e^{-\beta})_{r]\,k} \quad .$$

If we write the shear tensor $\sum_{mn}$ as

$$\sum_{mn} = S_{mn} - \frac{1}{3}\delta_{mn}\theta$$

and the expansion scalar θ as

$$\theta = S^{m}_{m} \quad ,$$

then the field equations for submanifolds of Bianchi I are:

$$3\theta^2 - \frac{1}{2}\sum_{mn}\sum_{mn} + \frac{1}{2}r = \rho \quad ,$$

$$\left((\theta\delta_{km}) + \sum_{km}\right) u_k t_{\ell m} + t_{\ell m}\dot{u}_m = q_\ell \quad ,$$

$$-6\dot{\theta} - 9\theta^2 - \frac{3}{2}\sum_{m\ell}\sum_{m\ell} - \frac{1}{2}r = 3p \quad ,$$

and

$$\dot{\sum}_{nk} + 3\theta \sum_{nk} + [\theta, t]_{nk} + r_{nk} - \frac{1}{3}\delta_{nk} r = \pi_{nk}$$

where $\dot{y} = \partial y/\partial x^o$; the spatial curvature r is caused by the $t_{ij}u_k$ term in the rotation coefficients; q_ℓ and π_{nk} are the momentum-density flux and trace-free stresses. The hydrodynamic equations are

$$\dot{\rho} + (\rho + p)\,\theta = 0$$

and

$$(\rho + p)\,\dot{u}_m = -h^n_m\, p_{|n} \quad .$$

The reader may have realized by now that the purpose of this discussion is to demonstrate the triviality of finding inhomogeneous solutions. Therefore, we immediately specialize to the seemingly most unpleasant case: Perfect fluid ($\pi_{nk} = q_\ell = 0$), diagonal β_{mn}, and dust ($p = 0$). Then, the acceleration equations read

$$a_m \equiv \dot{u}_m + u_\ell\,(\dot{\alpha}\,\delta_{\ell m} + \sigma_{\ell m}) = 0$$

which solve to

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$$u_m = u^o_m(\hat{x})\, e^{-\alpha - \beta_m}$$

where $u^o_m(\hat{x})$ is a function of the space coordinates. Now, α and β_m are functions of x^o, $\alpha(x^o)$, and $\beta_m(x^o)$. Let $\alpha = \ell n a$ and the Friedman-like equation is

$$da/dx^o = \left[\frac{\rho_o}{a^3} + \frac{\sigma_o^2}{2a^6}\right]^{1/2} a - u^o_{m|m}\, e^{-\beta_m} \quad .$$

Since $a = a(x^o)$, we are free to find a solution with $u^o_{m|m} = 0$. Then

$$u^o_m = \nabla_m \phi$$

where

$$\phi = \sum a_n O_n(\hat{x})$$

and a_n is a constant determined by the boundary conditions inserted into the usual Fourier-Bessel integrals

$$a_n = \int f(x)\, O_n(x)\, dx$$

and $O_n(\hat{x})$ is any set of orthogonal functions consistent with a chosen symmetry for the flat-space Laplacian implied by

$$\nabla^m u^o_m = \nabla^2 \phi = 0 \quad .$$

An implicit integral is easily found for $a(x^o)$. The explicit dependence of $\beta_m(x^o)$ is found from the shear equations.

C. Extended Locally Rotationally Symmetric Class III Cosmologies

Wainwright and Szafron [17,35,36] have extended the locally rotationally symmetric (LRS) universes [16] to inhomogeneous models of the Sekeres [37] type. They suggest a general adaptation of these techniques to study large classes of solutions to Einstein's equations with a dust source [17,36]. Rather than concentrate on a hack-type duplication of Class II results for Class III LRS (although we shall do so eventually), refer to Paragraphs V.A and V.B.

Finding outrageous, inhomogeneous solutions is not a very formidable task, as previously revealed in Paragraphs V.A and V.B. The arbitrariness allows any density's spatial dependence to presume spatial hypersurfaces. Therefore, inhomogeneous and single solutions are neither very challenging nor very informative.

Szafron, Wainwright, and Eardley have suggested a solution to the previously mentioned problem. Rather than groping solution-by-solution through Einstein's equations, group structures should instead be used to give an elegant classification to families of solutions and relate those families to each other mainly as each other's subsets in the group parameter space.

It seems that these inhomogeneous models are of little use individually. Also, the solutions seem too easy to obtain. An "easy" way out would be to compute the observational calculations in each case. Then, one would test for the allowed inhomogeneous cosmological models by comparison with observations. However, this procedure seems tedious and unrewarding.

It seems better to study the structure of various generalizations from the homogeneous cases to the inhomogeneous cases. One then can treat the problem as the consideration of successively larger classes of solutions. The main consideration is the alteration of the group structure of the submanifold isometric under the lower dimensional group. One wants to know what physics there is corresponding to the choice of the higher dimensional group of which the additional free parameters will allow characterization of the enveloping inhomogeneous models. The only available method would be the reverse of the Inonu-Wigner contraction [14].

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In outline, if we have a group of generators $I_{1\nu}$ with a particular algebra

$$[I_{1\nu}, I_{1\mu}] = C^{1k}_{1\nu, 1\mu} I_{1k}$$

and the group is not compact, it may be possible to find a larger-dimension group to which the $I_{1\nu}$ group is a continuous subgroup. We seek a group of generators $J_{1\nu}$ which are found from $I_{1\nu}$ by a singular transformation

$$J_{1\nu} = I_{1\nu} + \epsilon V_{\mu\nu} I^{\mu}_{1}$$

and

$$J_{2\nu} = \epsilon I_{2\nu}$$

where $V_{\mu\nu}$ is a matrix representation of the group and as $\epsilon \to 0$,

$$J_{1\nu} = I_{1\nu} \quad .$$

Then the larger algebra is

$$[J_{1\nu}, J_{1\mu}] = C^{1k}_{1\nu, 1\mu} J_{1k} + \epsilon^{-1} C^{2k}_{1\nu, 1\mu} J_{2k} = [I_{1\nu}, I_{1\mu}]$$

$$+ \epsilon \left[V_{\nu\nu'} \delta_{\mu\mu'} + \delta_{\nu\nu'} V_{\mu\mu'} + \epsilon V_{\nu\nu'} V_{\mu\mu'} \right] [I_{1\nu}, I_{1\mu}] \quad ,$$

and as $\epsilon \to 0$ we recover the subalgebra. There is as yet no elegant way of performing this reverse process. Possibly, examination of the group parameters rather than guessing at the representation $V_{\mu\nu}$ of the higher dimensional group might work. For this method

$$a_{1\nu} = b_{1\nu} + \epsilon V_{\nu\mu} b^{\mu}_{1} \quad ,$$

and

$$a_{2\nu} = \epsilon b_{2\nu} \quad .$$

These parameters are directly related to certain relative tensors which define the structure constants C^{i}_{jk} of the algebra [32]. At the same time they directly specify the vector fields in the manifold which generate the algebra.

Spero and Baierlain [19] have suggested a variational approach which may lead to techniques for solving this problem. They treat the inhomogeneous metric as possessing an approximate symmetry. However, the following question should be considered when applying this solution: Under what conditions is this a "subsymmetry" of a higher-dimensional symmetry characterizing the inhomogeneous model?

The Class III LRS models offer the most challenging testing ground and shall be considered in a future publication.

VI. HOMOGENEOUS COSMOLOGICAL MODEL IN YANG'S THEORY OF GRAVITATION

Yang [7] has presented a gauge gravitation theory which is a rederivation of Kilmister [38]. In this section, a cosmological solution is investigated.

A metric is chosen in the form

$$ds^2 = dt^2 - Adx^2 - Ax^2dy^2 - Ax^2\sin^2 ydz^2$$

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and implemented in the MACSYMA symbolic manipulation system as the matrix

$$\begin{bmatrix} -A & 8 & 8 & 8 \\ 8 & -AX^2 & 8 & 8 \\ 8 & 8 & -AX^2 \, SIN^2(Y) & 8 \\ 8 & 8 & 8 & 1 \end{bmatrix}$$

where A is a function of t and x.

The Kilmister-Yang equations $R_{i[j \| k]} = 0$ then have the following surviving four components ($\partial A/\partial t = A_T$, $\partial A/\partial X = A_X$)

$$(D5) \quad \Big[2A^2 A_T A_{XX} X^2 - 2A^2 A_{TX} A_X X^2 + 2A^4 A_{TTT} X^2 - 2A^3 A_T A_{TT} X^2$$

$$+ 6A^2 A_T A_X X - 4A^3 A_{TX} X \quad ,$$

$$4A^4 A_{XXX} X^3 - 14A^3 A_X A_{XX} X^3 + 9A^2 A_X^3 X^3 + 4A^4 A_{TT} A_X X^3 - 4A^5 A_{TTX} X^3$$

$$+ 8A^4 A_{XX} X^2 - 16A^3 A_X^2 X^2 - 8A^4 A_X X \quad ,$$

$$6A^3 A_T A_{XX} X^2 - 9A^2 A_T A_X^2 + 8A^3 A_{TX} A_X X^2 + 4A^5 A_{TTT} X^2 - 4A^4 A_T A_{TT} X^2$$

$$- 4A^4 A_{TXX} X^2 + 18A^3 A_T A_X X - 4A^4 A_{TX} X, \; 2A^4 A_{TTX} X^2 - 2A^3 A_{TT} A_X X^2 \quad ,$$

$$- \frac{2A^3 CX - 4AA_{XX} X + 3A_X^2 X + 6A^2 A_{TT} X - 8AA_X}{2A^3 X} \Big] \quad .$$

Assuming C is the constant scalar curvature, $A(t,_X)$ is separable, and $A = P(X)\,Q(t)$, the equations take the form

$$(D6)\quad \Bigg[2P^2Q^3X\left(P^3QQ_{TTT}X - P^3Q_TQ_{TT}X + PP_{XX}Q_TX - P_X^2Q_TX + PP_XQ_T\right) ,$$

$$P^2Q^5X\left(4P^2P_{XXX}X^2 - 14PP_XP_{XX}X^2 + 9P_X^3X^2 + 8P^2P_{XX}X - 16PP_X^2X - 8P^2P_X\right) ,$$

$$P^3Q^4X\left(4P^3QQ_{TTT}X - 4P^3Q_TQ_{TT}X + 2PP_{XX}Q_TX - P_X^2Q_TX + 6PP_XQ_T\right) , 8 ,$$

$$- \frac{6P^3Q_{TT}X + 2CP^3QX - 4PP_{XX}X + 3P_X^2X - 8PP_X}{2P^3QX} \Bigg] .$$

The second of the equations involves only the function $P(X)$. A solution is $P = X^{-4}$.

Upon substitution, the following equations for Q survive

$$\left[\frac{2Q^3(QQ_{TTT} - Q_TQ_{TT})}{X^{18}} , 8 , \frac{4Q^4(QQ_{TTT} - Q_TQ_{TT})}{X^{22}} , 8 , -\frac{3Q_{TT} + CQ}{Q} \right] ,$$

each component separately equal to zero.

The equation involving the curvature scalar is

$$Q_{TT} = -\frac{CQ}{3} .$$

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Upon differentiation, we find

$$Q_{TTT} = -\frac{CQ_T}{3} \quad ,$$

which identically satisfies the remaining field equations. Thus, we find

$$Q = Q_o \exp[\sqrt{-C/3}\, t] \quad .$$

The metric is then

$$ds^2 = dt^2 - Q_o e^{\sqrt{-C/3}\, t} X^{-4}(dX^2 + X^2 d\Omega^2)$$

which can be easily transformed to

$$ds^2 = dt^2 - Q_o \exp[\sqrt{-C/3}\, t]\, [dr^2 + r^2 d\Omega^2] \quad .$$

A similar metric has been found as a cosmological solution in a modified version of Yilmaz' theory [39]. It is also similar to the metric in the steady-state theory [3]. Therefore, we refer to the solution as the Yang-Yilmaz universe.

The value of the scalar curvature C determines the large-scale behavior of the solution. For $C > 0$ the universe is an oscillating one, and for $C < 0$ it expands forever. This is a solution which is not an Einstein space.

The observations in the Yang-Yilmaz universe for $C < 0$ would be very similar to those in the steady-state universe [3]. For $C > 0$ we have a curious mixture of flat spatial sections with a closed universe. For real solutions we can arrange so that

$$Q = Q_o \cos Kt$$

where

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$$K = \sqrt{-C/3} \quad .$$

Then the universe oscillates in volume with the evolutionary period

$$2\pi/K = 2\pi\sqrt{3}/\sqrt{-C} \quad .$$

Present observational evidence seems to indicate an open universe (the $C > 0$ case) [40]. Nevertheless, observations would seem to be inconsistent with either choice of C; and, of course, for $C = 0$ we have only Minkowski space.

VII. TOWARD FINDING THE ROTATING SOLUTIONS FOR GRAVITATION THEORIES

The gyroscope experiment [41] was proposed to test whether an induction field (caused by rotating sources) exists in gravitation. This experiment is considered the "Faraday" experiment of relativistic gravitation theories. In the weak-field limit, this test of the existence of such fields is essential. The existence of the binary pulsar allows extension of the gyroscope experiment's results to strong gravity fields; therefore, it may be of use in testing gravitation theories. To do this the fields exterior to rotating stars must be found for each viable gravitation theory. This may be posed as the problem of determining the axially symmetric, stationary asymptotically flat solutions for empty space for these gravity theories. Since most of these theories possess a Riemannian geometry, one possible technique would involve use of the Newman-Penrose identities in finding the solutions in which the gravitational field's directions of propagation are expanding and twisting. They may not be shear free, and, therefore, there may be no analogue of the Kerr [3] solution in Einstein's theory. An alternative and more meaningful approach might involve the use of the "Killing" vectors of the static spherically symmetric solution to extend to the "Killing" vectors of the stationary axially symmetric case as has been done in Einstein's theory [42]. Relying on that, one may make use of the tetrad vectors $E_{a\mu}$ from which a metric g_{ab} is composed,

$$g_{ab} = E_a^{\mu} E_{b\mu} \quad ,$$

and try to extend them.

The solutions for the static spherically symmetric case in the viable gravitation theories are in the form

$$ds^2 = -e^{2\phi} dt^2 + e^{2\psi} dR^2 + e^{2\psi} R^2 d\theta^2 + e^{2\psi} R^2 \sin^2\theta d\varphi^2$$

where

$$\phi = \phi(R)$$

and

$$\psi = \psi(R) \quad .$$

The appropriate basis of differential forms ω^{μ} are

$$\left(ds^2 = n_{\mu\nu} \omega^{\mu} \omega^{\nu} \right) \omega^{o} = e^{\phi} dt \quad ,$$

$$\omega^{1} = e^{\psi} dR \quad ,$$

$$\omega^{2} = e^{\psi} R d\theta \quad ,$$

and

$$\omega^{3} = e^{\psi} R \sin\theta d\varphi \quad .$$

We can immediately extend these forms to the stationary axially symmetric case by writing [43]

$$\omega^{o} = e^{\phi}\,[dt + f(\theta)\,d\varphi] \quad ,$$

$$\omega^{1} = e^{\psi} dR \quad ,$$

$$\omega^{2} = e^{\psi} R d\theta \quad ,$$

and

$$\omega^{3} = e^{\psi} R \sin\theta d\varphi \quad .$$

The contravariant basis dual to these forms is

$$\left(< \omega^{\mu}, e_{\nu} > = \delta^{\mu}_{\nu} \right), e_{o} = e^{-\phi} \partial/\partial t \quad ,$$

$$e_{1} = e^{-\psi} \partial/\partial R \quad ,$$

$$e_{2} = e^{-\psi} R^{-1} \partial/\partial\theta \quad ,$$

and

$$e_{3} = [e^{\psi} R \sin\theta]^{-1} \partial/\partial\varphi - 8(\theta)\ [e^{\psi} R \sin\theta]^{-1} e^{-\phi} \partial/\partial t \quad .$$

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The composition of the metric is

$$g_{\mu\nu} = E^{i}_{\mu} E_{\nu i} \left(\omega^{\mu} = E^{\mu}_{i}\, dx^{i} \right)$$

and

$$g^{\mu\nu} = J^{\mu i} J_i^{\nu} \left(e_{\mu} = J_{\mu}^{i} \, \partial / \partial x^{i} \right).$$

From this we have

$$E_o^o = e^{\phi} \quad ,$$

$$E_3^o = e^{\phi} f(\theta) \quad ,$$

$$E_1^1 = e^{\psi} \quad ,$$

$$E_2^2 = e^{\psi} R$$

and

$$E_3^3 = e^{\psi} R \sin\theta$$

for the covariant basis, and

$$J_o^o = e^{-\phi} \quad ,$$

$$J_1^1 = e^{-\psi} \quad ,$$

$$J_2^2 = e^{-\psi} / R \quad ,$$

$$J_3^3 = [e^{\psi} R \sin\theta]^{-1} \quad ,$$

and

$$J_o^3 = -f(\theta)^2/[e^{\psi} R \sin\theta] \quad .$$

It is this vector decomposition that should be inserted directly into the field equations of the particular gravity theory and the solutions found for the functions E_i^{μ} and J_{μ}^{i}. The Newman-Penrose identities can then be used to investigate the properties of the solutions. In view of the fact that for all the alternative theories of gravity $R_{\mu\nu} \neq 0$, it may not be possible to find the solution with shear-free expanding and twisting rays; hence, there is no analogue to the Kerr [44] or Newman-Tamburino-Unti [45] solutions in Einstein's theories. These solutions are presently being worked out, and it is expected that they will be obtained systematically and with facility.

The equivalent in matter-filled spaces would be shearing universes with twisting rays. These solutions also will be investigated using the algorithm previously mentioned.

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APPROVAL

GRAVITATION

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or nuclear energy activities or programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

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